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Graphs with maximum size and lower bounded girth[☆]

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ABSTRACT

For integers $n \geq 4$ and $\nu \geq n+1$, let $ex(\nu; \{C_3, \dots, C_n\})$ denote the maximum number of edges in a graph of order ν and girth at least $n+1$. The $\{C_3, \dots, C_n\}$-free graphs with order ν and size $ex(\nu; \{C_3, \dots, C_n\})$ are called extremal graphs and denoted by $EX(\nu; \{C_3, \dots, C_n\})$. We prove that given an integer $k \geq 0$, for each $n \geq 2\log_2(k+2)$ there exist extremal graphs with ν vertices, $\nu + k$ edges and minimum degree 1 or 2. Considering this idea we construct four infinite families of extremal graphs. We also see that minimal $(r; g)$-cages are the exclusive elements in $EX(\nu_0(r, g); \{C_3, \dots, C_{g-1}\})$.

1. Introduction

For undefined terminology and notation we refer the reader to [1]. Let $V(G)$ and $E(G)$ denote respectively the set of vertices and the set of edges of a graph G. The order of G is denoted by $\nu = \nu(G)$ and its size by $e = e(G)$. The *average degree* of a graph is $\bar{d} = \bar{d}(G) = 2e/n$. The cycle of length n, $n \geq 3$, is denoted by C_n. Recall that the *girth* of a graph G, $g = g(G)$, is the length of the shortest cycle in G and if G does not contain a cycle, we set $g(G) = \infty$.

This work deals with $\mathrm{ex}(\nu; \{C_3, \dots, C_n\})$, the maximum number of edges in a simple graph of order ν and girth at least $n+1$. We refer to it as the extremal function. By $EX(\nu; \{C_3, C_4, \dots, C_n\})$ we denote the set of all simple graphs of order ν, girth at least $n+1$, and with $\mathrm{ex}(\nu; \{C_3, C_4, \dots, C_n\})$ edges. Its elements are called extremal graphs.

It is well known (see [2]) that $\mathrm{ex}(\nu; \{C_3\}) = \lfloor \nu^2/4 \rfloor$. Hence, we assume throughout this work that $n \geq 4$. The values of $\mathrm{ex}(\nu; \{C_3, C_4\})$ for $\nu \leq 24$ are given in [3], but proofs of some of them appear in [4]. The corresponding ones for $25 \leq \nu \leq 30$ are determined in [5]. For $n \in \{5, 6, 7\}$ lower bounds are provided in [6,7] and some exact values in [8].

More general results establish that $\mathrm{ex}(2n+2; \{C_3, \dots, C_n\}) = 2n+4$ for $n \geq 12$ (see [9]) and that $\mathrm{ex}(\nu; \{C_3, \dots, C_n\}) = \nu$ for $n+1 \leq \nu \leq \lceil \frac{3n-1}{2} \rceil$ (see [10]).

Due to the difficulty that is involved in the determination of the exact value of the extremal function, general characteristics of the extremal graphs are studied in the papers [10,11,3,9]. In this work we first determine general properties of the extremal graphs. In particular we prove in Theorem 2 that under certain conditions, the extremal graphs are quite similar to one another. That is, many of them are simply obtained by subdividing the edges of a certain simple graph or a multigraph until the forbidden cycles are avoided and the girth becomes as large as desired.

Adopting this idea, we construct four infinite families of extremal graphs. To prove their extremality we make use of the next result obtained by Alon, et al. in [12]:

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Theorem A *([12]). For $g \geq 3$ and $d > 2$, put*

$$\nu_0(d,g) = \begin{cases} 1 + d\sum_{i=0}^{\frac{g-3}{2}}(d-1)^i & \text{if } g \text{ is odd} \\ 2\sum_{i=0}^{\frac{g-2}{2}}(d-1)^i & \text{if } g \text{ is even.} \end{cases}$$

A graph G with average degree $\bar{d}$ and girth g has at least $\nu_0(\bar{d}, g)$ vertices.

2. Results

Definition 1. Let $k \geq 0$ and $n \geq 4$ be integers. Let $G_k(n)$ denote the set of all graphs G such that $e(G) - \nu(G) = k$ and $g(G) > n$.

To prove that this set is not empty it is enough to consider the graph formed by $k+2$ independent paths of length $\lfloor n/2 \rfloor + 1$ with the same endvertices. It then makes sense to consider the minimum order of any graph in this set.

Definition 2. Let $k \geq 0$ be an integer. For each $n \geq 4$ we define

$$\nu_k(n) = \min\{\nu(G) : G \in G_k(n)\}.$$

It is clear that $\text{ex}(\nu_k(n); \{C_3, \ldots, C_n\}) \geq \nu_k(n)+k$ but both the equality and the inequality are really possible. If $k = n = 4$, it is known [3] that $\text{ex}(9; \{C_3, C_4\}) = 12$ and that $\text{ex}(10; \{C_3, C_4\}) = 15$. Therefore, $\nu_4(4) > 9$ and it is feasible to construct graphs in $G_4(4)$ with 10 vertices, for example, the graph obtained by removing an edge from the Petersen cage. This proves that $\nu_4(4) = \nu_5(4) = 10$, and that no graph of $G_4(4)$ is extremal.

Taking into account that the function $\text{ex}(\nu; \{C_3, \ldots, C_n\})$ increases strictly in ν, it is obvious that $\text{ex}(\nu; \{C_3, \ldots, C_n\}) \geq \nu + k$ for every $\nu \geq \nu_k(n)$. Consequently the knowledge of the numbers $\nu_k(n)$ and $\nu_{k+1}(n)$ determines the set of orders ν such that the function $\text{ex}(\nu; \{C_3, C_4, \ldots, C_n\}) - \nu$ is a constant.

Proposition 1. *Let $k \geq 0$ and $n \geq 4$ be integers such that $\nu_k(n) < \nu_{k+1}(n)$. Then,*

$$\{\nu : \text{ex}(\nu; \{C_3, \ldots, C_n\}) = \nu + k\} = \{\nu : \nu_k(n) \leq \nu < \nu_{k+1}(n)\}.$$

The importance of the above defined numbers $\nu_k(n)$ is such that they allow one to change the difficult problem of the determination of the exact value of the extremal function $\text{ex}(\nu; \{C_3, \ldots, C_n\})$ to an easier one. In fact, the knowledge of two consecutive numbers $\nu_k(n)$, $\nu_{k+1}(n)$ provides general results on the function $\text{ex}(\nu; \{C_3, \ldots, C_n\})$.

The graphs of $G_k(n)$ with minimum order $\nu_k(n)$ satisfy some particular properties.

Proposition 2. *Let $k \geq 0$ and $n \geq 4$ be integers. Every graph G in $G_k(n)$ with order $\nu_k(n)$ has minimum degree $\delta(G) \geq 2$ and girth $g(G) = n + 1$.*

Proof. Let G be a graph in the set $G_k(n)$ with $\nu_k(n)$ vertices.

The minimum degree of G must be at least 2, because otherwise there exists a vertex $u \in V(G)$ with degree $\delta(u) = 1$ and the graph $G' = G - u$ belongs to $G_k(n)$ and has number of vertices less than $\nu_k(n)$.

Likewise, if $g(G) = n + 2$, the graph G' obtained from G by contracting one of its edges belongs to $G_k(n)$ and satisfies $\nu(G') = \nu_k(n) - 1$. □

This result implies that any graph in $EX(\nu_k(n); \{C_3, \ldots, C_n\})$ has exactly girth $n + 1$, extending the ideas of the papers [10,11,3,9].

Returning on the values $\nu_k(n)$, we see next that they increase strictly in n but not necessarily strictly in k. Moreover, as a consequence of Proposition 1 we can affirm that there are at most $\lfloor n/2 \rfloor$ orders ν such that $\text{ex}(\nu; \{C_3, \ldots, C_n\}) - \nu$ remains constant.

Lemma 1. *For integers $k \geq 0$ and $n \geq 4$ the following hold:*

(i) $\nu_k(n) \leq \nu_{k+1}(n)$.
(ii) $\nu_{k+1}(n) \leq \nu_k(n) + \lfloor n/2 \rfloor$.

Proof. (i) By removing one edge from every graph in $G_{k+1}(n)$ one obtains a graph in $G_k(n)$ with the same order. This implies that $\nu_k(n) \leq \nu_{k+1}(n)$.

(ii) Let $G \in G_k(n)$ be a graph with order $\nu_k(n)$. Since $g(G) > n$, there exist u, v in G such that $d_G(u, v) \geq \lceil n/2 \rceil$. The graph G' obtained from G by adding a path of length $\lfloor n/2 \rfloor + 1$ joining the vertices u, v belongs to $G_{k+1}(n)$, and therefore, $\nu(G') = \nu_k(n) + \lfloor n/2 \rfloor \geq \nu_{k+1}(n)$. □

Although we have mentioned that the set $\{\nu : \nu_k(n) \le \nu < \nu_{k+1}(n)\}$ can be empty, this does not happen when $n \ge 2\log_2(k+2)$. Moreover, infinite extremal graphs G such that $e(G) - \nu(G) = k$ can be constructed by an adequate subdivision of edges until the forbidden cycles are avoided and the girth becomes as large as desired.

Theorem 2. *Let $k \ge 1$ be an integer. For all $n \ge 2\log_2(k+2)$ there exists ν such that:*

(i) $\mathrm{ex}(\nu; \{C_3, \dots, C_n\}) = \nu + k$.
(ii) *Every graph of $EX(\nu; \{C_3, \dots, C_n\})$ has minimum degree 1 or 2.*

Proof. First we prove that the existence of a graph $G \in \mathcal{G}_k(n)$ such that $\delta(G) \ge 3$ implies that $n < 2\log_2(k+1)$.

Let us assume that, for $k \ge 1$, there exists a graph G in $\mathcal{G}_k(n)$ with $\delta(G) \ge 3$ and maximum degree $\Delta(G) = \Delta$. Let us denote by n_i the number of vertices of degree i in G. Obviously, $n_3 + n_4 + \cdots + n_\Delta = \nu(G)$ and $3n_3 + 4n_4 + \cdots + \Delta n_\Delta = 2\,e(G) = 2(\nu(G) + k)$.

We conclude that

$$\nu(G) = n_3 + n_4 + \cdots + n_\Delta \le n_3 + 2n_4 + \cdots + (\Delta - 2)n_\Delta = 2k.$$

On the other hand, since $\bar{d}(G) \ge \delta(G) \ge 3$ and $g(G) \ge n + 1$, it follows from Theorem A that

$$\nu(G) \ge \begin{cases} 3 \cdot 2^{n/2} - 2 & \text{if } n \text{ is even} \\ 2 \cdot (2^{(n+1)/2} - 1) & \text{if } n \text{ is odd.} \end{cases}$$

Both inequalities concerning $\nu(G)$ imply that $n < 2\log_2(k+1)$.

(i) For given integers $k \ge 0$ and $n \ge 2\log_2(k+2)$, let $G \in \mathcal{G}_{k+1}(n)$ be any extremal graph with order $\nu_{k+1}(n)$. As we have just seen, and according to Proposition 2, G has a vertex x such that $\delta(x) = 2$. The graph $G' = G - x$ belongs to $\mathcal{G}_k(n)$ and therefore $\nu_{k+1}(n) - 1 = \nu(G) \ge \nu_k(n)$. This inequality and Proposition 1 imply that at least for $\nu = \nu_k(n)$ the equality $\mathrm{ex}(\nu; \{C_3, \dots, C_n\}) = \nu + k$ holds.
(ii) Given $k \ge 0$ and $n \ge 2\log_2(k+2)$ the existence of an order ν such that $\mathrm{ex}(\nu; \{C_3, \dots, C_n\}) = \nu + k$ is guaranteed from the previous remark. Then every graph $G \in EX(\nu; \{C_3, \dots, C_n\})$ belongs to $\mathcal{G}_k(n)$ and its minimum degree $\delta(G)$ must be 1 or 2, because otherwise the assumption concerning n is contradicted as we have proved before. □

In particular, by convenient subdivisions we present below four infinite families of extremal graphs.

Theorem 3. *For $n \ge 4$ and $k \in \{1, 5, 7, 15\}$, we have*

$$\mathrm{ex}(a_k(n); \{C_3, \dots, C_n\}) = a_k(n) + k,$$

where the values of $a_k(n)$ are determined as follows:

(i) *if $n \ge 4$, $a_1(n) = \lceil (3n+1)/2 \rceil$;*
(ii) *if $n \ne 6$, $a_5(n) = 3n - 2$;*
(iii) *if $n \ne 6, 7, 8, 9, 10, 12, 16, 18$,*

$$a_7(n) = \begin{cases} \lfloor (7n-4)/2 \rfloor & \text{if } n \equiv 1, 2, 3 \pmod 6 \\ (7n-2)/2 & \text{if } n \equiv 0, 4 \pmod 6 \\ (7n-7)/2 & \text{if } n \equiv 5 \pmod 6; \end{cases}$$

(iv)

$$a_{15}(n) = \begin{cases} \lfloor (45n-34)/8 \rfloor & \text{if } n \ge 274 \text{ and } n \equiv 0, 1, 2, 4, 6 \pmod 8 \\ \lfloor (45n-55)/8 \rfloor & \text{if } n \ge 155 \text{ and } n \equiv 3, 5 \pmod 8 \\ (45n-75)/8 & \text{if } n \ge 7 \text{ and } n \equiv 7 \pmod 8). \end{cases}$$

Proof. For given integers $n \ge 4$ and $k \in \{1, 5, 7, 15\}$, we consider the graph displayed in Fig. 1, taking into account that every edge is subdivided the indicated number of times according to its color. Since this graph belongs to the set $\mathcal{G}_k(n)$ and has order $a_k(n)$, it follows that $\mathrm{ex}(a_k(n); \{C_3, \dots, C_n\}) \ge a_k(n) + k$.

The another inequality comes from the result in [12], since the assumption of the existence of graphs with girth at least $n + 1$, order $a_k(n)$ and size strictly greater than $a_k(n) + k$ contradicts Theorem A. □

If we extend the definition of $a_k(n)$ to being the order of the graphs in Fig. 1, then we can also affirm that $\mathrm{ex}(a_k(n); \{C_3, \dots, C_n\}) \ge a_k(n) + k$ for the excluded values of n.

Notice that we have just constructed a family of extremal graphs by subdivisions of the multigraph formed by two vertices joined by three edges. In contrast, subdivisions of pseudographs can be avoided whenever the extremal graphs contain more than one cycle.

Proposition 3. *Let $n \ge 4$ and $\nu \ge \lceil (3n+1)/2 \rceil$ be integers. No graph in $EX(\nu; \{C_3, \dots, C_n\})$ can be obtained by subdivisions of a pseudograph.*

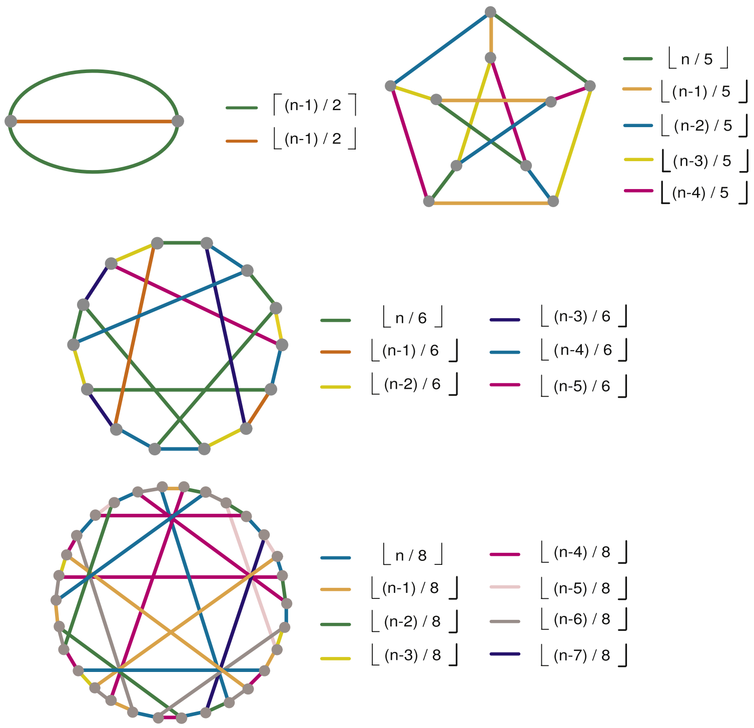

Fig. 1. Four infinite families of extremal graphs.

Proof. Assume that there exists an extremal $\{C_3, \ldots, C_n\}$-free graph G, with order $\nu(G) \geq \lceil (3n+1)/2 \rceil$, which is a subdivision of a pseudograph. If we write $k = e(G) - \nu(G)$, from Theorem 3 it follows that $k \geq 1$. Clearly, G contains one cycle with m, $(m \geq n)$ consecutive vertices, $x_1, x_2, \ldots, x_m$, with degree 2. The graph $G' = G - \{x_1, \ldots, x_m\}$ belongs to $\mathcal{G}_{k-1}(n)$, and consequently, $\nu(G') \geq \nu_{k-1}(n)$. From Proposition 1 it is known that $\nu_{k+1}(n) > \nu(G) = \nu(G') + m \geq \nu(G') + n \geq \nu_{k-1}(n) + n$. This inequality contradicts Lemma 1(ii). □

Theorem 2 and Proposition 3 let us conclude that some extremal graphs are not very different from each other, because many of them are simply obtained by subdividing the edges of a graph or multigraph until the forbidden cycles are avoided. Once the girth becomes as large as desired and the order ν is such that $\nu_k(n) < \nu < \nu_{k+1}(n)$, the extremal graphs can also contain paths appended to some vertices of a cycle.

It is known (see [3]) that the Petersen and the Hoffman–Singleton cages are the unique elements in $EX(10; \{C_3, C_4\})$ and $EX(50; \{C_3, C_4\})$ respectively. This result can be extended, and in fact minimal $(r; g)$-cages are the exclusive elements in $EX(\nu_0(r, g); \{C_3, \ldots, C_{g-1}\})$. To prove this, we take into consideration that the authors of [12] define $\Lambda(G) = \prod_{u \in V(G)} (\delta(u) - 1)^{\frac{\delta(u)}{2e(G)}}$ and affirm that any graph G with girth g verifies

$$\nu(G) \geq \nu_0(\Lambda(G) + 1, g) \geq \nu_0(\bar{d}(G), g). \tag{1}$$

Taking into account that $F(x) = x \log(x-1)$ is a convex function and consequently for any $x_0 \geq 2$ the inequality $F(x) \geq F(x_0) + F'(x_0)(x - x_0)$ holds, it can be proved that $\Lambda(G) + 1 = \bar{d}(G)$ if and only if G is a regular graph.

Theorem 4. *Let $r \geq 3$ and $g \geq 5$ be given integers. If there exists a minimal $(r; g)$-cage with order $\nu_0(r; g)$, then*

$$EX(\nu_0(r; g); \{C_3, \ldots, C_{g-1}\}) = \{\textit{minimal } (r; g)\textit{-cages}\}.$$

Proof. Let us assume that $r \geq 3$ and $g \geq 5$ are integers such that $(r; g)$-cages are minimal with order $\nu_0(r; g)$. First, to prove that these graphs belong to the family $EX(\nu_0(r; g); \{C_3, \ldots, C_{g-1}\})$, it is enough to combine Theorem A and the strictly increasing function $\nu_0(d, g)$.

Second, any non-regular graph $G \in EX(\nu_0(r; g); \{C_3, \ldots, C_{g-1}\})$ satisfies $\Lambda(G) + 1 > \bar{d}(G) = r$ and contradicts inequality (1). □

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